THE NUMBER OF GROUP HOMOMORPHISMS FROM D_m INTO D_n

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Abstract. We derive general formulæ for counting the number of homomorphisms between dihedral groups using only elementary group theory.

This note considers the problem of counting the number of group homomorphisms from D_m into D_n, where for a positive integer l, D_l denotes the finite group generated by two generators r_l and f_l subject to the relations $r_l^l = e = f_l^2$ and $r_l f_l = f_l r_l^{-1}$. We derive some general formulæ using only elementary group theory and a few basic facts about the dihedral groups. We will assume throughout that ϕ represents Euler's totient function.

Theorem 1. *Let m and n be positive odd integers. The number of group homomorphisms from D_m into D_n is*

$$1 + n\left(\sum_{k|\gcd(m,n)} \phi(k)\right). \tag{1}$$

Proof. Suppose that $\rho\colon D_m \to D_n$ is a group homomorphism, where m and n are positive odd integers. We consider all of the places that ρ could send the generators r_m and f_m of D_m which yield group homomorphisms. As m is odd, it must be the case that $\rho(r_m) = r_n^\alpha$, where r_n^α is an element of D_n whose order divides both m and n. Let k represent the order of this element. There are precisely $\phi(k)$ elements of order k in D_n. Since ρ can send r_m to any one of these elements, we have $\sum_{k|m,n} \phi(k)$ choices for $\rho(r_m)$.

Next, consider our choices for $\rho(f_m)$. Since $|\rho(f_m)|$ divides $|f_m| = 2$, either $\rho(f_m) = r_n^\beta f_n$, $0 \le \beta < n$, or $\rho(f_m) = e_n$. But not all of these choices for $\rho(f_m)$ yield homomorphisms, as can be seen when we consider where ρ sends the remaining elements in D_m of the form $r_m^k f_m$, where $0 < k < m$. If $\rho(f_m) = e_n$ and $\rho(r_m) = r_n^\alpha$, where $\alpha \ne 0$ or n, then $\rho(r_m f_m) = r_n^\alpha e_n = r_n^\alpha$, and $|r_n^\alpha|$ does not divide $|r_m f_m|$. Therefore, if $\rho(f_m) = e_n$, then ρ must be trivial. Conversely, when $\rho(f_m) = r_n^\beta f_n$, $\rho(r_m^k f_m) = r_n^{k\alpha+\beta \mod n} f_n$, and $|r_n^{k\alpha+\beta \mod n} f_n|$ divides $|r_m^k f_m|$. So, given any choice for r_m, we have n choices for f_m. Including the trivial homomorphism gives the result. □

When m and n are positive odd integers and $m|n$, it follows from the fact that $\sum_{k|n} \phi(k) = n$ [1] that there are $mn + 1$ group homomorphisms from D_m into D_n, and furthermore, there are $n^2 + 1$ group endomorphisms of D_n.

When m is a positive odd integer and n is a positive even integer, $r_n^{n/2}$ is a possible choice for the image of f_m. However, if f_m is sent to $r_n^{n/2}$, then the image of r_m must be e_n; otherwise the map fails to be a homomorphism. Again let $\rho\colon D_m \to D_n$ denote the map and suppose that $\rho(r_m f_m) = r_n^\alpha r_n^{n/2}$ for some $\alpha \ne 0$

or n This element necessarily has order not equal to 2 or 1; a contradiction. So in this case, we gain a single additional map sending r_m to e_n and f_m to $r_n^{n/2}$. Taking this additional consideration into account, a proof nearly identical to that used for Theorem 1 yields the following result.

Theorem 2. *Let m be a positive odd integer and n a positive even integer. The number of group homomorphisms from D_m into D_n is*

$$2 + n \left(\sum_{k|\ \gcd(m,n)} \phi(k) \right). \tag{2}$$

When m is a positive even integer, the number of choices that exist for the image of r_m includes all elements of the form $r_n^k f_n$, $0 \leq k < n$. This creates a number of additional possibilities.

Theorem 3. *Let m and n be positive even integers. The number of group homomorphisms from D_m into D_n is*

$$4 + 4n + n \left(\sum_{k|\ \gcd(m,n)} \phi(k) \right). \tag{3}$$

Proof. Suppose that $\rho \colon D_m \to D_n$ is a group homomorphism, where m and n are positive even integers. When m is even, we have in addition to the $\sum_{k|m,n} \phi(k)$ possible choices for $\rho(r_m)$ that occur when m is odd the possibility of mapping r_m to those elements in D_n of the form $r_n^\beta f_n$. As there are n such elements of the latter type, we have $\sum_{k|m,n} \phi(k) + n$ possible choices for $\rho(r_m)$.

Next, suppose $\rho(r_m) = r_n^\alpha$ and consider $\rho(f_m)$. Since $|\rho(f_m)|$ divides $|f_m| = 2$, it must be the case that either $\rho(f_m) = r_n^\beta f_n$, $0 \leq \beta < n$, $\rho(f_m) = r_n^{n/2}$, or $\rho(f_m) = e_n$. If $\alpha = 0$ or $n/2$, any of these $n + 2$ choices for $\rho(f_m)$ will yield a homomorphism. If $\alpha \neq 0$ or $n/2$, then $\rho(f_m)$ cannot equal e_n or $r_n^{n/2}$. So, there are $n \left(\sum_{k|\ \gcd(m,n)} \phi(k) \right) + 4$ homomorphisms sending r_m to an element of the form r_n^α.

Assume next that $\rho(r_m) = r_n^\alpha f_n$. Since $|\rho(r_m)| = |\rho(f_m)| = 2$, it follows that if ρ is a homomorphism, then the size of the image of ρ is either 2 or 4. There is only one subgroup of each order containing $r_n^\alpha f_m$; the cyclic subgroup $\langle r_n^\alpha f_m \rangle$, and the subgroup $\langle r_n^\alpha f_m, r_n^{\alpha + n/2 \mod n} f_n \rangle$. There are two choices for f_m which result in the first case; namely, $\rho(f_m) = e_n$, or $\rho(f_m) = r_n^\alpha$. Similarly, there are two choices for f_m which result in the second case; $\rho(f_m) = r_n^{\alpha + n/2 \mod n} f_n$ or $\rho(f_m) = r^{n/2}$. A brief calculation shows that each of these four possibilities does in fact give a homomorphism, which leads to the conclusion. □

When m and n are positive even integers and $m|n$, it follows that the number of group homomorphisms from D_m into D_n is $4 + 4n + mn$, while the number of group endomorphisms of D_n is $(n + 2)^2$.

The last case to consider is when m is even and n is odd.

Theorem 4. *Let m be a positive even integer and n a positive odd integer. The number of group homomorphisms from D_m into D_n is*

$$1 + 2n + n \left(\sum_{k|\ \gcd(m,n)} \phi(k) \right). \tag{4}$$

Proof. As in the proof of Theorem 1, there are $n\left(\sum_{k\mid \gcd(m,n)} \phi(k)\right)$ homomorphisms in which r_m is sent to an element of the form r_n^α, $0 < \alpha < n$, plus the trivial homomorphism. In addition, we could send r_m to any of the n elements of the form $r_n^\alpha f_n$, $0 \le \alpha < n$. If $\rho(r_m) = r_n^\alpha f_n$, then the image of ρ is a subgroup of order 2, the cyclic subgroup $\langle \rho(r_m) \rangle$. That leaves two choices for $\rho(f_m)$; either $\rho(f_m) = e_n$ or $\rho(f_m) = r_n^\alpha f_n$, from which the result follows. □

When $\gcd(m,n) = 1$, Theorems 2 and 4 lead to the succinct formulæ that the number of group homomorphisms from D_m into D_n equals $n+2$ when m is odd and n is even, and $3n+1$ when m is even and n is odd.

References

[1] W. Sierpinski, *Elementary Theory of Numbers*, 2nd ed., North-Holland, Amsterdam.

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